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THE LINEAR ELASTIC COSSERAT SURFACE AND SHELL THEORY

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Abstract—The linearized theory of an elastic Cosserat surface is discussed emphasizing its relevance to the classical problem of the linear theory of elastic shells (and plates) regarded as three dimensional bodies.

1. INTRODUCTION AND NOTATION

A GENERAL theory of a Cosserat surface has been given by Green *et al.* [1], and various developments of this theory have been discussed by Green and Naghdi [2–5]. A related theory, valid only for elastic surfaces, has been considered by Cohen and DeSilva [6]. Here we confine attention to a linear elastic Cosserat surface and our main purpose is to underline the relevance of this theory to the classical problem of linear elastic shells regarded as three dimensional bodies.

We summarize the main notation and formulae for a *linear* elastic Cosserat surface and refer readers to the paper by Green *et al.* [1] for full details. Let x^α, $(\alpha = 1, 2)$, be a system of two curvilinear coordinates on the undeformed and unstressed initial two dimensional surface embedded in a Euclidean 3-space; and let $\mathbf{A}_\alpha$, $\mathbf{A}^\alpha$ be covariant and contravariant base vectors associated with x^α, and $\mathbf{A}_3$ a unit normal vector to the surface. Throughout the paper, Greek indices have the value 1, 2 and Latin indices the values 1, 2, 3. The metric tensors for the surface are $A_{\alpha\beta}$, $A^{\alpha\beta}$ and $B_{\alpha\beta}$ is the second fundamental form. Associated with each point of the surface is a director $\mathbf{D}$, a function of x^α, and

$$\mathbf{D} = D_i\mathbf{A}^i = D^i\mathbf{A}_i. \tag{1.1}$$

The (infinitesimal) displacement vector of the surface is $\mathbf{u}$ and the (infinitesimal) director displacement is $\boldsymbol{\delta}^*$, where

$$\mathbf{u} = u_i\mathbf{A}^i = u^i\mathbf{A}_i, \qquad \boldsymbol{\delta}^* = \delta_i^*\mathbf{A}^i = \delta^{*i}\mathbf{A}_i. \tag{1.2}$$

For later use, we also define the vector δ_i by

$$\begin{aligned} \delta_\alpha &= \delta_\alpha^* + D_\lambda(u^\lambda|_\alpha - B_\alpha^\lambda u_3) + D_3(u_{3,\alpha} + B_\alpha^\lambda u_\lambda), \\ \delta_3 &= \delta_3^* - D^\alpha(u_{3,\alpha} + B_\alpha^\lambda u_\lambda), \end{aligned} \tag{1.3}$$

where a comma denotes partial differentiation with respect to x^α and a vertical line denotes covariant differentiation with respect to the initial surface. In addition, we require the kinematic variables $e_{\alpha\beta}$, $\kappa_{\alpha\beta}$, $\kappa_{3\alpha}$ defined by

$$e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha|\beta}+u_{\beta|\alpha})-B_{\alpha\beta}u_3, \tag{1.4}$$

$$\kappa_{\alpha\beta} = \delta^*_{\alpha|\beta}-B_{\alpha\beta}\delta^*_3+\Lambda_{\lambda\beta}(u^\lambda{}_{|\alpha}-B^\lambda_\alpha u_3)+\Lambda_{3\beta}(u_{3,\alpha}+B^\lambda_\alpha u_\lambda), \tag{1.5}$$

$$\kappa_{3\alpha} = \delta^*_{3,\alpha}+B^\lambda_\alpha\delta^*_\lambda-\Lambda^\lambda{}_{\cdot\alpha}(u_{3,\lambda}+B^\nu_\lambda u_\nu), \tag{1.6}$$

where

$$\Lambda_{\alpha\beta} = D_{\alpha|\beta}-B_{\alpha\beta}D_3, \qquad \Lambda_{3\alpha} = D_{3,\alpha}+B^\lambda_\alpha D_\lambda, \qquad \Lambda^\alpha{}_{\cdot\beta} = A^{\alpha\lambda}\Lambda_{\lambda\beta}. \tag{1.7}$$

The basic equations of motion for forces and director forces are†

$$\begin{aligned} N^{\alpha\beta}{}_{|\alpha}-B^\beta_\alpha N^{\alpha 3}+\rho F^\beta &= \rho c^\beta, \\ N^{\alpha 3}{}_{|\alpha}+B_{\alpha\beta}N^{\alpha\beta}+\rho F^3 &= \rho c^3, \end{aligned} \tag{1.8}$$

$$\begin{aligned} M^{\alpha\beta}{}_{|\alpha}-B^\beta_\alpha M^{\alpha 3}+\rho \bar{L}^\beta &= m^\beta, \\ M^{\alpha 3}{}_{|\alpha}+B_{\alpha\beta}M^{\alpha\beta}+\rho \bar{L}^3 &= m^3, \end{aligned} \tag{1.9}$$

and

$$N^{\alpha 3}+m^3D^\alpha-m^\alpha D^3+M^{\lambda 3}\Lambda^\alpha{}_{\cdot\lambda}-M^{\lambda\alpha}\Lambda_{3\lambda} = 0 \tag{1.10}$$

$$N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta}-m^\alpha D^\beta-M^{\lambda\alpha}\Lambda^\beta{}_{\cdot\lambda}. \tag{1.11}$$

In these formulae, ρ is the initial mass density per unit area of the surface, F^i and c^i are the assigned force and acceleration terms and $\bar{L}^i$ are the difference between the assigned director force and director inertia terms. The quantities m^i are specified by constitutive equations and (1.9) then represent equations of motion for director forces.

For an elastic Cosserat surface the Helmholtz free energy function is specified by

$$A = A(T, e_{\alpha\beta}, \kappa_{i\alpha}, \delta_i, A_{\alpha\beta}, \Lambda_{i\alpha}, D_i), \tag{1.12}$$

where T is temperature difference from some constant temperature of the initial surface. Moreover A is a quadratic function of T, $e_{\alpha\beta}$, $\kappa_{i\alpha}$, δ_i (with no linear terms) and the coefficients in this quadratic function may depend on $A_{\alpha\beta}$, $\Lambda_{i\alpha}$ and D_i. Also

$$S = -\frac{\partial A}{\partial T}, \tag{1.13}$$

$$N'^{\alpha\beta} = \rho\frac{\partial A}{\partial e_{\alpha\beta}}, \qquad m^i = \rho\frac{\partial A}{\partial \delta_i}, \qquad M^{\alpha i} = \rho\frac{\partial A}{\partial \kappa_{i\alpha}}, \tag{1.14}$$

where $\partial A/\partial e_{\alpha\beta}$ stands for $\frac{1}{2}[(\partial A/\partial e_{\alpha\beta})+(\partial A/\partial e_{\beta\alpha})]$ and S is entropy per unit mass.

2. SPECIAL COSSERAT SURFACE

We consider a special case of the foregoing theory when the initial director is along the normal to the initial surface and of constant length. Thus

$$D_\alpha = 0, \qquad D_3 = 1. \tag{2.1}$$

† There is a change of notation from that used by Green *et al.* [1]. The order of indices in $N^{\alpha i}$ and $M^{\alpha i}$ of the present paper corresponds to the usual notation in shell theory.

Hence, from (1.3) and (1.5)–(1.7), we have

$$\Lambda_{\alpha\beta} = -B_{\alpha\beta}, \qquad \Lambda_{3\alpha} = 0, \qquad \kappa_{\alpha\beta} = \rho_{\alpha\beta} - B_{\alpha\beta}\,\delta_3, \qquad \kappa_{3\alpha} = \rho_{3\alpha} + B^{\lambda}_{\alpha}\,\delta_{\lambda}, \tag{2.2}$$

where

$$-\rho_{\alpha\beta} = u_{3|\alpha\beta} + B^{\lambda}_{\alpha|\beta}u_{\lambda} + B^{\lambda}_{\alpha}u_{\lambda|\beta} + B^{\lambda}_{\beta}u_{\lambda|\alpha} - B_{\alpha\lambda}B^{\lambda}_{\beta}u_3 - \delta_{\alpha|\beta}, \qquad \rho_{3\alpha} = \delta_{3,\alpha}. \tag{2.3}$$

Also we may now write A in the different form

$$A = A^*(T, e_{\alpha\beta}, \rho_{i\alpha}, \delta_i, A_{\alpha\beta}, B_{\alpha\beta}). \tag{2.4}$$

Moreover, we then have

$$N'^{\alpha\beta} = \rho\frac{\partial A^*}{\partial e_{\alpha\beta}}, \qquad M^{\alpha i} = \rho\frac{\partial A^*}{\partial \rho_{i\alpha}}, \qquad V^i = \rho\frac{\partial A^*}{\partial \delta_i}, \tag{2.5}$$

where

$$\begin{aligned} V^{\beta} &= m^{\beta} + B^{\beta}_{\alpha}M^{\alpha 3} = M^{\alpha\beta}{}_{|\alpha} + \rho\bar{L}^{\beta},\\ V^3 &= m^3 - B_{\alpha\beta}M^{\alpha\beta} = M^{\alpha 3}{}_{|\alpha} + \rho\bar{L}^3. \end{aligned} \tag{2.6}$$

Also, from (1.10), (1.11) and (2.6),

$$N^{\alpha 3} = V^{\alpha}, \qquad N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta} + B^{\beta}_{\lambda}M^{\lambda\alpha}. \tag{2.7}$$

The basic equations of the theory are now (1.8) and (2.4)–(2.7).

In the above special theory of a Cosserat surface, the basic equations involve the kinematic variables $e_{\alpha\beta}$, $\rho_{i\alpha}$ and δ_i. These variables (which also occur in the energy equation) follow quite directly from (1.4)–(1.6) under the conditions (2.1). In the rest of this section, we discuss briefly an alternative development of the basic equations of the theory with a slightly different choice of the kinematic variables. For this purpose, put

$$\kappa_{\alpha\beta} = \bar{\rho}_{\alpha\beta} - B^{\lambda}_{\beta}e_{\lambda\alpha} - B_{\alpha\beta}\delta_3, \qquad \kappa_{3\alpha} = \bar{\rho}_{3\alpha} + B^{\lambda}_{\alpha}\delta_{\lambda}, \tag{2.8}$$

where†

$$\bar{\rho}_{\alpha\beta} = \rho_{\alpha\beta} + B^{\lambda}_{\beta}e_{\lambda\alpha}, \qquad \bar{\rho}_{3\alpha} = \delta_{3,\alpha} \tag{2.9}$$

$$\begin{aligned} \bar{\rho}_{(\alpha\beta)} = {}& \tfrac{1}{2}(\delta_{\alpha|\beta} + \delta_{\beta|\alpha}) - [u_{3|\alpha\beta} + B^{\lambda}_{\alpha|\beta}u_{\lambda} + B^{\lambda}_{\alpha}u_{\lambda|\beta} + B^{\lambda}_{\beta}u_{\lambda|\alpha} - B_{\alpha\lambda}B^{\lambda}_{\beta}u_3]\\ & + \tfrac{1}{2}(B^{\lambda}_{\alpha}e_{\lambda\beta} + B^{\lambda}_{\beta}e_{\lambda\alpha}), \end{aligned} \tag{2.10}$$

$$\bar{\rho}_{[\alpha\beta]} = \tfrac{1}{2}(\delta_{\alpha|\beta} - \delta_{\beta|\alpha}) + \tfrac{1}{2}(B^{\lambda}_{\beta}e_{\lambda\alpha} - B^{\lambda}_{\alpha}e_{\lambda\beta}), \tag{2.11}$$

and $\rho_{\alpha\beta}$ is given by $(2.3)_1$. Assuming now for the free energy the form

$$A = \bar{A}(T, e_{\alpha\beta}, \bar{\rho}_{i\alpha}, \delta_i, A_{\alpha\beta}, B_{\alpha\beta}), \tag{2.12}$$

we can then deduce the results

$$N^{(\alpha\beta)} = \rho\frac{\partial \bar{A}}{\partial e_{\alpha\beta}}, \qquad M^{\alpha i} = \rho\frac{\partial \bar{A}}{\partial \bar{\rho}_{i\alpha}}, \qquad V^i = \rho\frac{\partial \bar{A}}{\partial \delta_i}, \tag{2.13}$$

where

$$N^{(\alpha\beta)} = \tfrac{1}{2}(N^{\alpha\beta} + N^{\beta\alpha}) = N'^{\alpha\beta} - \tfrac{1}{2}[M^{\lambda\alpha}B^{\beta}_{\lambda} + M^{\lambda\beta}B^{\alpha}_{\lambda}]. \tag{2.14}$$

† The notation $\bar{\rho}_{3\alpha}$ in $(2.8)_2$ and $(2.9)_2$ is introduced only for later convenience, even though $\bar{\rho}_{3\alpha} = \rho_{3\alpha}$.

Also,

$$N^{[\alpha\beta]} = \tfrac{1}{2}(N^{\alpha\beta} - N^{\beta\alpha}) = \tfrac{1}{2}(B^{\alpha}_{\lambda}M^{\lambda\beta} - B^{\beta}_{\lambda}M^{\lambda\alpha}). \tag{2.15}$$

The basic equations of this alternative form of the theory are (1.8), (2.6), $(2.7)_1$ and (2.12)–(2.15).

3. PLATE THEORY

The form of the strain energy for the linear theory of a Cosserat plate has been given by Green and Naghdi [2]; this form imitates the symmetry properties of a three dimensional plate which is transversely isotropic with respect to normals to the plate. Restricting attention here to isothermal deformations, the corresponding value of A in the notation of the previous section is†

$$\begin{aligned} 2\rho A^* = {} & [\alpha_1 A^{\alpha\beta}A^{\gamma\delta} + \alpha_2(A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})]e_{\alpha\beta}e_{\gamma\delta} + \alpha_3 A^{\alpha\beta}\delta_\alpha\delta_\beta + \alpha_4\delta_3^2 \\ & + [\alpha_5 A^{\alpha\beta}A^{\gamma\delta} + \alpha_6 A^{\alpha\gamma}A^{\beta\delta} + \alpha_7 A^{\alpha\delta}A^{\beta\gamma}]\rho_{\alpha\beta}\rho_{\gamma\delta} + \alpha_8 A^{\alpha\beta}\rho_{3\alpha}\rho_{3\beta} + 2\alpha_9 A^{\alpha\beta}e_{\alpha\beta}\delta_3, \end{aligned} \tag{3.1}$$

where

$$\rho_{\alpha\beta} = \delta_{\alpha|\beta} - u_{3|\alpha\beta}, \qquad \rho_{3\alpha} = \delta_{3,\alpha}. \tag{3.2}$$

Green and Naghdi [2] showed that the corresponding equations separated into two groups, one for extensional theory and one for bending theory of a plate.

The coefficients $\alpha_1, \dots \alpha_9$ in (3.1) are so far arbitrary. We now concentrate our attention on relating the present theory to the three dimensional theory of isotropic plates which have constant thickness h and we identify $N^{\alpha\beta}$, $N^{\alpha 3}$ as stress resultants and shear stress resultants, and $M^{\alpha\beta}$, $M^{\alpha 3}$ as stress couples and shear stress couples, respectively. Also, V^3 can be identified as a normal stress resultant. By considering the problem of simple flexure of a plate for which an exact three dimensional solution exists, Green and Naghdi [2] make the identification

$$\alpha_6 + \alpha_7 = (1-\nu)B, \qquad \alpha_5 = \nu B, \qquad B = \frac{Eh^3}{12(1-\nu^2)}, \tag{3.3}$$

where E is Young's modulus and ν Poisson's ratio for an isotropic plate. Moreover, since in three dimensional plate theory $M^{\alpha\beta}$ is symmetric, we take

$$\alpha_6 = \alpha_7. \tag{3.4}$$

We emphasize, however, that there may well be conditions for which the identification (3.4) is undesirable. The only remaining coefficient concerned with bending of the plate is α_3 and this remains arbitrary at present.

Using the values (3.3) and (3.4), the stress–strain relations for the bending of a plate become

$$M^{\alpha\beta} = M^{\beta\alpha} = BH^{\alpha\beta\gamma\delta}\rho_{\gamma\delta}, \qquad V^\alpha = \alpha_3 A^{\alpha\gamma}\delta_\gamma, \tag{3.5}$$

† The notation in this Section is patterned after the form of the theory of Cosserat surface characterized by the equations (1.8) and (2.4)–(2.7). In the case of a flat plate ($B_{\alpha\beta} = 0$), the two forms of the theory in Section 2 become identical.

where

$$H^{\alpha\beta\gamma\delta} = \tfrac{1}{2}\{A^{\alpha\gamma}A^{\beta\delta}+A^{\alpha\delta}A^{\beta\gamma}+\nu[2A^{\alpha\beta}A^{\gamma\delta}-A^{\alpha\gamma}A^{\beta\delta}-A^{\alpha\delta}A^{\beta\gamma}]\}$$
$$= \tfrac{1}{2}\{A^{\alpha\gamma}A^{\beta\delta}+A^{\alpha\delta}A^{\beta\gamma}+\nu(\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}+\epsilon^{\alpha\delta}\epsilon^{\beta\gamma})\}. \tag{3.6}$$

In (3.6), $\epsilon^{\alpha\beta}$ is an ϵ-symbol, a contravariant tensor.

The five coefficients $\alpha_1, \alpha_2, \alpha_4, \alpha_8, \alpha_9$ are concerned with extensional theory and in order to relate their values with three dimensional isotropic elastic coefficients we record the stress–strain relations

$$N'^{\alpha\beta} = N^{\alpha\beta} = N^{\beta\alpha} = [\alpha_1 A^{\alpha\beta}A^{\gamma\delta}+\alpha_2(A^{\alpha\gamma}A^{\beta\delta}+A^{\alpha\delta}A^{\beta\gamma})]e_{\gamma\delta}+\alpha_9 A^{\alpha\beta}\delta_3,$$
$$V^3 = \alpha_4\delta_3+\alpha_9 A^{\alpha\beta}e_{\alpha\beta}, \qquad M^{\alpha 3} = \alpha_8 A^{\alpha\gamma}\rho_{3\gamma}. \tag{3.7}$$

Referred to rectangular Cartesian axes x_i, with x_3 normal to the plane of the plate, the stresses $\sigma_{\alpha\beta}, \sigma_{33}$ and three dimensional strains $\gamma_{\alpha\beta}, \gamma_{33}$ are related by the equations

$$\sigma_{\alpha\beta} = \frac{D(1-\nu)}{2h}\left[\frac{2\nu}{1-2\nu}\gamma_{\lambda\lambda}\delta_{\alpha\beta}+2\gamma_{\alpha\beta}\right]+\frac{D\nu(1-\nu)}{h(1-2\nu)}\gamma_{33},$$
$$\sigma_{33} = \frac{D(1-\nu)^2}{h(1-2\nu)}\gamma_{33}+\frac{D\nu(1-\nu)}{h(1-2\nu)}\gamma_{\lambda\lambda}, \qquad D = \frac{Eh}{1-\nu^2}, \tag{3.8}$$

where

$$\gamma_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial v_\alpha}{\partial x_\beta}+\frac{\partial v_\beta}{\partial x_\alpha}\right), \qquad \gamma_{33} = \frac{\partial v_3}{\partial x_3}, \tag{3.9}$$

v_α, v_3 being three dimensional components of displacements. If we integrate equations (3.8) with respect to x_3 through the thickness of the plate, we obtain equations of the same form as (3.7) with

$$\frac{1}{h}\int_{-h/2}^{h/2} v_\alpha \,\mathrm{d}x_3 \quad \text{and} \quad \frac{1}{h}[v_3]_{-h/2}^{h/2}$$

corresponding, respectively, with u_α and δ_3. We therefore make the identification

$$\alpha_1 = \frac{\nu(1-\nu)}{1-2\nu}D, \qquad \alpha_2 = \frac{1-\nu}{2}D,$$
$$\alpha_4 = \frac{(1-\nu)^2}{1-2\nu}D, \qquad \alpha_9 = \frac{\nu(1-\nu)}{1-2\nu}D, \tag{3.10}$$

where D is defined by $(3.8)_3$. It does not appear to be possible to identify α_8 which therefore remains arbitrary. Using (3.10), equations (3.7) become

$$N^{\alpha\beta} = N^{\beta\alpha} = DH^{\alpha\beta\gamma\delta}e_{\gamma\delta}+\frac{\nu}{1-\nu}A^{\alpha\beta}V^3,$$
$$V^3 = D\frac{(1-\nu)^2}{1-2\nu}\delta_3+D\frac{\nu(1-\nu)}{1-2\nu}A^{\alpha\beta}e_{\alpha\beta}, \tag{3.11}$$
$$M^{\alpha 3} = \alpha_8 A^{\alpha\gamma}\rho_{3\gamma}.$$

We observe that the arbitrary coefficients α_3 and α_8 have the orders of magnitude

$$\alpha_3 = O(D), \qquad \alpha_8 = O(B). \tag{3.12}$$

4. ISOTROPIC SHELLS

With a view toward isotropic elastic shells, we discuss in this Section the linear theory of a Cosserat surface under the condition (2.1). Although we confine our attention mainly to the form of the theory characterized by the basic equations (1.8) and (2.4)–(2.7), a parallel discussion can be presented for shells using the alternative form of theory of Cosserat surface characterized by (1.8), (2.6), $(2.7)_1$ and (2.12)–(2.15). As is evident from (2.4), in the isothermal case, the Helmholtz free energy function A^* depends on $e_{\alpha\beta}, \rho_{i\alpha}, \delta_i$ (and $A^{\alpha\beta}$) with $\rho_{i\alpha}$ defined by (2.3), as well as on $B_{\alpha\beta}$. For an isotropic Cosserat surface (with a center of symmetry) A^* is an isotropic function of its arguments. If, in addition, it imitates the properties of a three dimensional shell which is transversely isotropic with respect to normals to the middle surface, then A^* is unaltered by the transformation

$$\delta_\alpha \to -\delta_\alpha, \qquad u_3 \to -u_3, \qquad \delta_3 \to \delta_3, \qquad u_\alpha \to u_\alpha, \qquad B_{\alpha\beta} \to -B_{\alpha\beta}, \tag{4.1}$$

so that

$$A^*(e_{\alpha\beta}, \rho_{\alpha\beta}, \rho_{3\alpha}, \delta_\alpha, \delta_3, A_{\alpha\beta}, B_{\alpha\beta}) = A^*(e_{\alpha\beta}, -\rho_{\alpha\beta}, \rho_{3\alpha}, -\delta_\alpha, \delta_3, A_{\alpha\beta}, -B_{\alpha\beta}). \tag{4.2}$$

With A^* also an isotropic function, it is now possible to write down the complete form for A^* which is quadratic in $e_{\alpha\beta}, \rho_{i\alpha}, \delta_i$. The expression will contain terms of the form (3.1), with $\rho_{i\alpha}$ defined by (2.3), together with terms involving $B_{\alpha\beta}$. These latter terms vanish for a plate so that we can still make the identification (3.3), (3.4), (3.10) and (3.12) for the coefficients in (3.1). However, the complete expression for A^* including terms involving $B_{\alpha\beta}$ is rather unwieldly, and the basic problem for shells is what choice to make for A^*. It should be emphasized that this problem is exactly parallel to the problem one has in choosing a suitable strain energy function in the three dimensional nonlinear or linear elasticity and is a common feature of all general theories. Any special choice of A^* can be regarded as a special case of the general theory and *not* as an approximation.

Suppose coordinates on the surface are chosen so that at a point $A_{\alpha\beta}, e_{\alpha\beta}, h\rho_{i\alpha}, \delta_i$ are of zero dimensions, where $e_{\alpha\beta}$ and $\rho_{i\alpha}$ are given by (1.4) and (2.3), respectively. Also $B_{\alpha\beta}$ is of dimension R^{-1}, where R is a typical radius of curvature of the surface. We assume that $e_{\alpha\beta}, h\rho_{i\alpha}, \delta_i$ are all of the same order of magnitude so that the terms in (3.1) for a shell are all $O(D)$. Terms which contain $B_{\alpha\beta}$ and which complete the strain energy expression for a shell will be

$$O\left[D\left(\frac{h}{R}\right)^n\right], \tag{4.3}$$

where $n = 1, 2, \ldots$ Even if, in addition to (3.1), we only include all terms $O[D(h/R)]$, we shall have a large number of extra coefficients to determine.

Here, for simplicity, we restrict our attention to the special Cosserat surface whose free energy function is specified by (3.1). If we regard this as imitating a three dimensional shell then this means that we are neglecting all terms of $O(h/R)$ or smaller compared with those in (3.1); but, again we emphasize that this latter interpretation is not essential.† The stress–strain relations for shells are then

$$
\begin{aligned}
N'^{\alpha\beta} &= N'^{\beta\alpha} = DH^{\alpha\beta\gamma\delta}e_{\gamma\delta}+\frac{\nu}{1-\nu}A^{\alpha\beta}V^3, \qquad D = \frac{Eh}{1-\nu^2}\\
M^{\alpha\beta} &= M^{\beta\alpha} = BH^{\alpha\beta\gamma\delta}\rho_{\gamma\delta}, \qquad B = \frac{Eh^3}{12(1-\nu^2)}\\
N^{\alpha 3} &= V^{\alpha} = \alpha_3 A^{\alpha\gamma}\delta_{\gamma},\\
V^3 &= D\frac{(1-\nu)^2}{1-2\nu}\delta_3 + D\frac{\nu(1-\nu)}{1-2\nu}A^{\alpha\beta}e_{\alpha\beta},\\
M^{\alpha 3} &= \alpha_8 A^{\alpha\gamma}\rho_{3\gamma},\\
N'^{\alpha\beta} &= N'^{\beta\alpha} = N^{\alpha\beta}+M^{\lambda\alpha}B^{\beta}_{\lambda},
\end{aligned}
\tag{4.4}
$$

where $e_{\alpha\beta}$, $\rho_{i\alpha}$ are defined by (1.4) and (2.3). The remaining equations of the theory are (1.8) and (2.6).

It is now a straightforward matter to obtain shell equations of a classical type from the above theory. Let

$$\frac{\alpha_8}{\alpha_1} \to 0, \qquad \frac{\alpha_3}{\alpha_5} \to \infty, \qquad \delta_\alpha \to 0, \tag{4.5}$$

in such a way that

$$V^\alpha \to \text{finite limit}, \qquad M^{\alpha 3} \to 0, \tag{4.6}$$

and V^α is not determined by a constitutive equation. Also, if the director body force and inertia terms are either zero or negligible, then $\bar{L}^3 = 0$, $\bar{L}^\beta = 0$ and, from (2.6) and (4.6),

$$V^3 = 0, \qquad M^{\alpha\beta}{}_{|\alpha} = V^\alpha = N^{\alpha 3}, \tag{4.7}$$

equations of motion (1.8) being unchanged. In view of $(4.7)_1$, equation $(4.4)_1$ reduces to

$$N'^{\alpha\beta} = N'^{\beta\alpha} = DH^{\alpha\beta\gamma\delta}e_{\gamma\delta}, \tag{4.8}$$

and $(4.4)_4$ yields

$$\delta_3 = -\frac{\nu}{1-\nu}A^{\alpha\beta}e_{\alpha\beta}. \tag{4.9}$$

The remaining equations $(4.4)_2$ and $(4.4)_6$ are unchanged but now, from (2.3) and (4.5),

$$-\rho_{\alpha\beta} = -\rho_{\beta\alpha} = u_{3|\alpha\beta} + B^{\lambda}_{\alpha|\beta}u_\lambda + B^{\lambda}_{\alpha}u_{\lambda|\beta} + B^{\lambda}_{\beta}u_{\lambda|\alpha} - B_{\alpha\beta}B^{\lambda}_{\beta}u_3. \tag{4.10}$$

This reduction to a classical (interior) type theory for shells fits in with the reduction indicated by Green and Naghdi [3] for nonlinear shell theory. There will be a corresponding reduction in the number of boundary conditions and these reduced boundary conditions can be shown to be of the usual form [4].

† If we adopt this interpretation, then it affects our view of equations such as (2.7); see the discussion at the end of this section.

The linear classical type of equations given by (4.7)–(4.10), $(4.4)_2$, $(4.4)_6$ and (1.8) are of the same form as those given by Naghdi [7]. Because of the approximations involved in obtaining classical type shell equations from the three dimensional equations of elasticity, the equations of Naghdi can be shown to be equivalent, within the order of approximation involved, to those obtained by Koiter [8] and by Green and Naghdi [9]†. The same equivalence can also be seen if we adopt the view, considered at the beginning of this section, to the effect that in the strain–energy function we are neglecting all terms $O(h/R)$ or smaller compared with those in (3.1). It then follows that in such expressions as $(4.4)_6$ which yield (2.14), i.e.,

$$N^{(\alpha\beta)} = \tfrac{1}{2}(N^{\alpha\beta}+N^{\beta\alpha}) = N'^{\alpha\beta}-\tfrac{1}{2}M^{\lambda\alpha}B^{\beta}_{\lambda}-\tfrac{1}{2}M^{\lambda\beta}B^{\alpha}_{\lambda},$$

we can omit the second and third terms and write, to the same order of approximation,

$$N^{(\alpha\beta)} = N'^{\alpha\beta}. \tag{4.11}$$

We thus recover the form given by Green and Naghdi [9]. With a suitable choice of the free energy function $\bar{A}$ in (2.12), the equations equivalent to those in [8] and [10] can also be recovered from the exact equations of the alternative theory in Sec. 2, namely (1.8), (2.6), $(2.7)_1$ and (2.12)–(2.15). We must, of course, retain (2.15) for all forms of the theory.

To conclude, we recall that equations (1.4), (1.8), (2.3), (2.6) and (4.4) form an exact system of equations of a Cosserat surface in which the strain energy function has been chosen to have the special form (3.1). We adopt this system as suitable for discussing the main properties of three dimensional isotropic thin shells. Associated with this system of equations, we have six boundary conditions on each boundary of the shell.

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†See also [10].